A COMBINATORIAL APPROACH TO JUMPING PARTICLES

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Abstract. In this paper we consider a model of particles jumping on a row of cells, called in physics the one dimensional totally asymmetric exclusion process (TASEP). More precisely we deal with the TASEP with open or periodic boundary conditions and with two or three types of particles. From the point of view of combinatorics a remarkable feature of this Markov chain is that it involves Catalan numbers in several entries of its stationary distribution.

We give a combinatorial interpretation and a simple proof of these observations. In doing this we reveal a second row of cells, which is used by particles to travel backward. As a byproduct we also obtain an interpretation of the occurrence of the Brownian excursion in the description of the density of particles on a long row of cells.

1. Jumping particles

We shall consider a model of jumping particles on a row of n cells that was exactly solved in the early 90's in physics, under the name *one dimensional totally asymmetric exclusion process with open boundaries*, or TASEP for short. Roughly speaking, the TASEP consists of black particles entering a row of n cells from an infinite reservoir on the left-hand side and randomly hopping to the right with the simple exclusion rule that each cell may contain at most one particle.

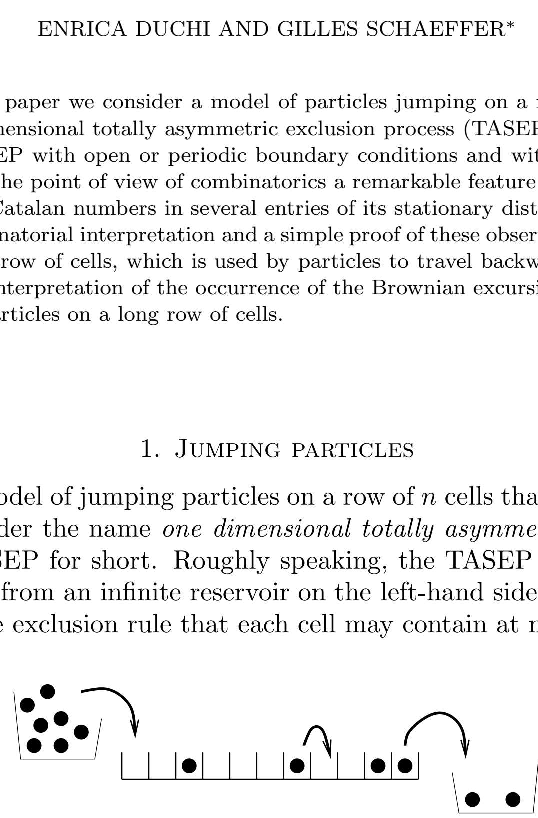

Figure 1. An informal illustration of the TASEP.

The TASEP is usually defined as a continuous-time Markov process on a finite set of configurations of particles on a line. We shall use an alternative definition as a finite state Markov chain —with discrete time— which is more convenient for our combinatorial purpose. One could insist on calling our chain the TASEC, with "C" for chain instead of "P" for process, but as we will argue later, there is no need for this distinction. Another cosmetic modification we allow ourselves consists in putting a white particle in each empty cell, so as to make explicit the left-right particle-hole symmetry of the chain.

1.1. Definition of the TASEP.

A *TASEP configuration* is a row of n cells, each containing either one black particle or one white particle (see Figure 2). These cells are delimited by $n+1$ walls: the left border (or wall 0), the ith separation wall for $i = 1, \dots, n-1$, and the right border (or wall n).

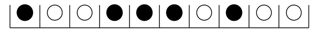

Figure 2. A basic configuration with $n = 10$ cells.

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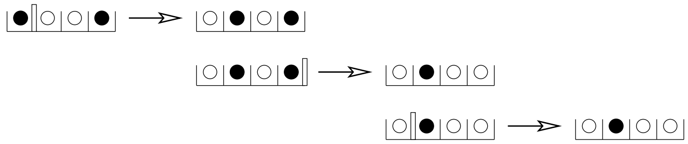

FIGURE 3. An example of an evolution, with $n = 4$ and $\alpha = \beta = \gamma = 1$. The active wall triggering each transition is indicated by the symbol $\|$.

The TASEP is a Markov chain $S^0_{\alpha\beta\gamma}$ defined on the set of TASEP configurations for any three parameters α, β and γ in the interval $]0, 1]$. From time t to $t + 1$, the chain evolves from the configuration $\tau = S^0_{\alpha\beta\gamma}(t)$ to a configuration $\tau' = S^0_{\alpha\beta\gamma}(t + 1)$ as follows:

- A wall i is chosen uniformly at random among the $n + 1$ walls, and then may become *active* with probability $\lambda(i)$, with $\lambda(i) = \alpha$ for $i = 1, \dots, n - 1$, $\lambda(0) = \beta$ and $\lambda(n) = \gamma$.
- If the wall does not become active, then nothing happens: $\tau' = \tau$.
- Otherwise from τ to τ' some changes may occur near the active wall:
  a. If the active wall is not a border ($i \in \{1, \dots, n - 1\}$) and has a black particle on its left-hand side and a white one on its right-hand side, then these two particles swap: $\bullet\|\circ \to \circ|\bullet$.
  b. If the active wall is the left border ($i = 0$) and the leftmost cell contains a white particle, then this particle leaves the system and is replaced by a black one: $\|\circ \to |\bullet$.
  c. If the active wall is the right border ($i = n$) and the rightmost cell contains a black particle, then this particle leaves the system and is replaced by a white one: $\bullet\| \to \circ|$.
  d. Otherwise the configuration is left unchanged: $\tau' = \tau$.

As illustrated by Figure 3, black particles travel from left to right and white particles do the opposite. The entire chain for $n = 3$ is shown in Figure 11. The four cases a, b, c, d define an application $\vartheta : (\tau, i) \mapsto \tau'$ from the set of configurations with an active wall into the set of configurations. The definition of the TASEP can be rephrased in terms of this application as: at time t choose a random wall $i = I(t)$ and set

$$S^0_{\alpha\beta\gamma}(t+1) = \begin{cases} \vartheta(S^0_{\alpha\beta\gamma}(t), i) & \text{with probability } \lambda(i), \\ S^0_{\alpha\beta\gamma}(t) & \text{otherwise.} \end{cases}$$

The parameters α, β and γ control the rate at which particles try to move inside the system and at the borders. In particular, we shall call *maximal flow regime* the special case $\alpha = \beta = \gamma = 1$, in which the rate at which particles try to move is maximal, and denote $S^0 = S^0_{111}$ the corresponding chain.

1.2. **Continuous-time descriptions of the TASEP.** In the physics literature, the TASEP is usually described in the following terms. The time is continuous, and one considers each wall independently: during any small time interval dt, wall i has probability $\lambda(i)dt$ to trigger a move $\omega \mapsto \vartheta(\omega, i)$. The rate $\lambda(i)$ takes the same values α, β, γ as previously.

Following the probabilistic literature [8], one can give a formulation which is equivalent to the previous one, but already closer to ours. In this description, each wall waits for an independent exponential random time with rate 1 before waking up (in other terms, at any time, the probability that wall i will still be sleeping after t seconds is e^{-t}). When wall i wakes up, it has probability $\lambda(i)$ to become active. If this is the case, then the transition $\omega \to \vartheta(\omega, i)$ is applied to the current configuration ω. In any case the wall falls again asleep, restarting its clock.

This continuous-time TASEP is now easily coupled to the Markov chain $S^0_{\alpha\beta\gamma}$. Let the time steps of $S^0_{\alpha\beta\gamma}$ correspond to the succession of moments at which a wall wakes up. Then in both versions, the index of the next wall to wake up is at any time a uniform random variable on $\{0, \dots, n\}$, and when a wall wakes up the transition probabilities are identical. This implies that the stationary distributions of the continuous-time TASEP and its Markov chain replica are the same.

1.3. **A remarkable stationary distribution.** Among many results on the TASEP, Derrida *et al.* [3, 5] proved the following nice properties of the chain $S^0 = S^0_{111}$, in which particles enter, travel and exit at the same maximal rate. First,

$$\text{Prob}(S^0(t) \text{ contains 0 black particles}) \underset{t\to\infty}{\longrightarrow} \frac{1}{C_{n+1}}, \tag{1.1}$$

where $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$ is the $(n+1)$th Catalan number. More generally, for all $0 \leq k \leq n$,

$$\text{Prob}(S^0(t) \text{ contains } k \text{ black particles}) \underset{t\to\infty}{\longrightarrow} \frac{\frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{n-k}}{C_{n+1}}, \tag{1.2}$$

where the numerators are called Narayana numbers.

The model is a finite state Markov chain which is clearly ergodic so that the previous limits are in fact the probabilities of the same events in the unique stationary distribution of the chain [7]. More generally, Derrida *et al.* provided expressions for the stationary probabilities of the chain $S^0_{\alpha\beta\gamma}$ for generic α, β, γ. Since their original work a number of papers have appeared providing alternative proofs and further results on correlations, time evolutions, etc. Recent advances and a bibliography can be found for instance in the article [6]. General books about this kind of particle processes are [8, 10].

However, the remarkable appearance of Catalan numbers in the stationary distribution of S^0 is not easily understood from the proofs in the physics literature. As far as we know, these proofs rely either on a *matrix ansatz*, or on a *Bethe ansatz*, both being then proved by a recursion on n.

1.4. **Combinatorial results.** Our main ingredient to study the TASEP consists in the construction of a new Markov chain $X^0_{\alpha\beta\gamma}$ on a set Ω^0_n of *complete configurations* that satisfies two main requirements: on the one hand the stationary distribution of the chain $S^0_{\alpha\beta\gamma}$ can be simply expressed in terms of that of the chain $X^0_{\alpha\beta\gamma}$; on the other hand the stationary behavior of the chain $X^0_{\alpha\beta\gamma}$ is easy to understand.

The complete configurations that we introduce for this purpose are made of two rows of n cells containing black and white particles. The first requirement is met by imposing that, disregarding what happens in the second row, the chain $X^0_{\alpha\beta\gamma}$ simulates in its first row the chain $S^0_{\alpha\beta\gamma}$. As illustrated by Figure 4, the second row will be used by black and white particles to return to their start point, thus revealing a circulation of the particles. The second requirement is met by adequately choosing the complete configurations and the transition rules so that $X^0_{\alpha\beta\gamma}$ has a simple stationary distribution: in particular in the case $\alpha = \beta = \gamma = 1$, $X^0 = X^0_{111}$ will have a uniform stationary distribution.

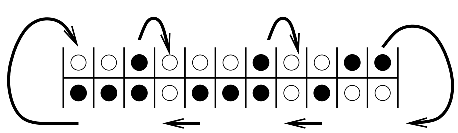

FIGURE 4. The circulation of black particles in the complete chain.

The chain $X^0_{\alpha\beta\gamma}$ is described in Section 2, together with a fundamental property of its transition rules. Our main result, presented in Section 3, is the combinatorial interpretation of the stationary distribution of the chain $S^0_{\alpha\beta\gamma}$, and in particular of Formulas (1.1)–(1.2).

It is known in the literature that some of the results on the TASEP can be extended to models with three particle types [1, 3]. We show that this is the case of Formulas (1.1)–(1.2) by adapting our approach in Section 4 to the 3-TASEP, a Markov chain $S_{\alpha\beta\gamma\varepsilon}$ in which there are 3 types of particles, $\bullet$, $\times$ and $\circ$ and transitions of the form

$$\bullet\|\times \to \times|\bullet, \qquad \bullet\|\circ \to \circ|\bullet, \quad \text{and} \quad \times\|\circ \to \circ|\times.$$

FIGURE 5. A complete configuration with $n = 10$, and a pair of rows violating the positivity condition at wall 4.

Our main results for the 3-TASEP are obtained by a relatively simple modification of the complete chain. In particular our combinatorial approach yields the following variant of Formula (1.1)–(1.2) for the chain $S = S_{111\frac{1}{2}}$: for any $k + \ell + m = n$,

$$\mathrm{Prob}(S(t) \text{ contains } \; k \bullet, \ell \times, \text{and } m \circ) \underset{t\to\infty}{\longrightarrow} \frac{\frac{\ell+1}{n+1}\binom{n+1}{k}\binom{n+1}{m}}{\frac{1}{2}\binom{2n+2}{n+1}}.$$

The TASEP and 3-TASEP are sometimes also defined with periodic boundary conditions: instead of giving special rules for walls 0 and n, one identifies these two walls and applies the same rule to every wall. With these boundary conditions, the stationary distribution of the TASEP is easily seen to be uniform. In Section 5 we apply our method to study the more interesting distribution of the 3-TASEP with periodic boundary conditions. In this chain the number of particles of each type is fixed (since they cannot leave the system), and, for $k \bullet$, $\ell \times$, $m \circ$, with $k + \ell + m = n$, we recover the known result:

$$\mathrm{Prob}(\widehat{S}(t) \; = \; |\underbrace{\times\cdots\times}_{\ell}|\underbrace{\circ\cdots\circ}_{m}|\underbrace{\bullet\cdots\bullet}_{k}|) \underset{t\to\infty}{\longrightarrow} \frac{1}{\binom{n}{k}\binom{n}{m}}.$$

A different combinatorial proof of this later formula was recently proposed by Omer Angel [2].

2. The complete chain

2.1. **Complete configurations.** A *complete configuration* of Ω_n^0 is a pair of rows of n cells satisfying the following constraints:

(i) *The balance condition*: The two rows contain together n black and n white particles.
(ii) *The positivity condition*: On the left of any vertical wall there are at least as many black particles as white ones.

An example of complete configuration is given in Figure 5, together with a pair of rows that violates the positivity condition.

Given a complete configuration of length n, and an integer j, $0 \le j \le n$, let $B(j)$ and $W(j)$ be respectively the numbers of black and white particles lying in the first jth columns (from left to right), and set $E(j) = B(j) - W(j)$. In other terms, the quantities $B(j)$, $W(j)$ and $E(j)$ represent the number of black particles, the number of white particles, and their difference on the left of wall j. In particular, $E(0) = E(n) = 0$, and Condition (ii) of the definition of complete configurations reads $E(j) \ge 0$ for $j = 0, \ldots, n$ (this is why we call it a positivity condition). Readers with background in enumerative combinatorics may have recognized here complete configurations with n columns as bicolored Motzkin paths with n steps, or Dyck paths with $2n + 2$ steps in disguise [11, Chap. 6]. In particular these characterizations yield the following lemmas. A direct proof of these lemmas is given in Section 7 for completeness.

Lemma 2.1. *The number $|\Omega_n^0|$ of complete configurations is $C_{n+1} = \frac{1}{n+1}\binom{2n+2}{n} = \frac{1}{n+2}\binom{2n+2}{n+1}$.*

Lemma 2.2. *Let k, m, n be non negative integers with $k+m = n$. The number $|\Omega_{k,m}^0|$ of complete configurations of Ω_n^0 with k black and m white particles on the top row, and m black and k white particles on the bottom row is $\frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{m}$.*

A first hint to our interest in complete configurations should follow from the comparison of the lemmas with the probabilities in (1.1) and (1.2).

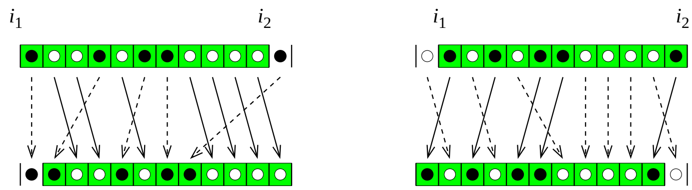

FIGURE 6. A white sweep and a black sweep.

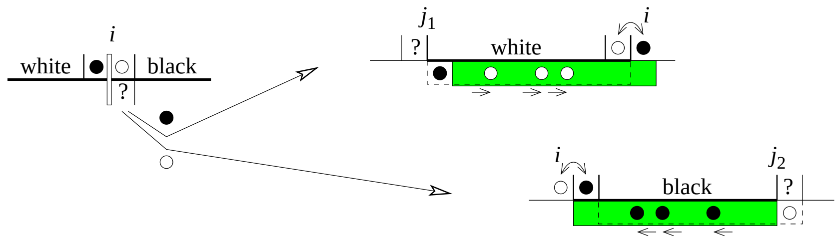

FIGURE 7. Sweeps occurring below the transition $(\bullet\|\circ \to \circ|\bullet)$.

2.2. **First definition of the complete chain.** The Markov chain $X^0_{\alpha\beta\gamma}$ on Ω^0_n will be defined in terms of an application T from the set $\Omega^0_n \times \{0,\dots,n\}$ to the set Ω^0_n that extends the application ϑ. Given a complete configuration ω and an active wall i, the action of T on the top row of ω does not depend on the second row, and mimics the action of ϑ as defined by cases a, b, c and d of the description of the TASEP. In particular in the top row, black particles travel from left to right and white particles from right to left. As opposed to that, in the bottom row, T moves black and white particles backward, as illustrated by Figure 4. In order to describe how these moves are performed, we first introduce the concept of sweep (see Figure 6):

- A *white sweep* between walls i_1 and i_2 consists in all white particles that are in the bottom row between walls i_1 and i_2 simultaneously hopping to the right (some black particles thus being displaced to the left in order to fill the gaps). For well definiteness a white sweep between i_1 and i_2 can occur only if the particle on the right-hand side of i_2 is black.
- A *black sweep* between walls i_1 and i_2 consists in all black particles that are in the bottom row between walls i_1 and i_2 simultaneously hopping to the left (some white particles thus being displaced to the right in order to fill the gaps). For well definiteness a white sweep between i_1 and i_2 can occur only if the particle on the left-hand side of i_1 is white.

Next, given a complete configuration and a wall i, we distinguish the following walls: if there is a black particle on the left-hand side of wall i in the top row, let $j_1 < i$ be the leftmost wall such that there are only white particles in the top row between walls j_1 and $i-1$; if there is a white particle on the right-hand side of i in the top row, let $j_2 > i$ be the rightmost wall such that there are only black particles in the top row between walls $i+1$ and j_2.

With these definitions, we are in the position to describe the action of T. Given a complete configuration $\omega \in \Omega^0_n$ and a wall $i \in \{0,\dots,n\}$, the cases a, b, c and d of the transition rule ϑ describe the top row of the image $T(\omega,i)$, and they are complemented as follows to describe the bottom row of the image:

a. In this case $i \in \{1,\dots,n-1\}$ and this wall separates a black and a white particle in the top row of ω. The moves in the bottom row then depend on the particle on the bottom right of wall i in ω: if it is black, a white sweep occurs between j_1 and i, otherwise it is white and a black sweep occurs between $i+1$ and j_2+1 (or between $i+1$ and n if $j_2 = n$). These moves are illustrated by Figure 7 (see also Figures 9–10, left and middle).

b. In this case $i = 0$ and the leftmost particle of the top row of ω is white. Then the leftmost column of ω is a $|{}^{\circ}_{\bullet}|$-column. These two particles exchange (so that a black particle enters in the top row in agreement with rule b for ϑ), and a black sweep occurs between the left

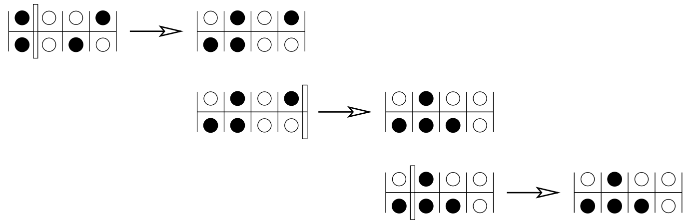

FIGURE 8. An example of actual evolution with $n = 4$ and $\alpha = \beta = \gamma = 1$.

border and wall $j_2 + 1$, or between the left and right borders if $j_2 = n$ (see Figure 10, right).

c. In this case $i = n$ and the rightmost particle of the top row of ω is black. Then the rightmost column of ω is a $|{}^{\bullet}_{\circ}|$-column. These two particles exchange (so that a white particle enters in the top row in agreement with rule c for ϑ), and a white sweep occurs between wall j_1 and the right border (see Figure 9, right).

d. Otherwise nothing happens.

The fact that the configuration $T(\omega, i)$ produced in each case satisfies the positivity constraint is not difficult to prove and it is explicitly checked in the next section.

The Markov chain $X^0_{\alpha\beta\gamma}$ on the set Ω^0_n of complete configurations with length n is defined from T exactly as the TASEP is described from ϑ: the evolution rule from time t to $t+1$ consists in choosing $i = I(t)$ uniformly at random in $\{0, \ldots, n\}$ and setting

$$X^0(t+1) = \begin{cases} T\left(X^0(t), i\right) & \text{with probability } \lambda(i), \\ X^0(t) & \text{otherwise.} \end{cases}$$

By construction, the Markov chains $S^0_{\alpha\beta\gamma}$ and $X^0_{\alpha\beta\gamma}$ are related by

$$S^0_{\alpha\beta\gamma} \equiv \operatorname{top}(X^0_{\alpha\beta\gamma}),$$

where $\operatorname{top}(\omega)$ denotes the top row of a complete configuration ω, and the $\equiv$ is intended as identity in law at any time, provided $S^0_{\alpha\beta\gamma}(0)$ and $\operatorname{top}(X^0_{\alpha\beta\gamma}(0))$ are equally distributed.

An appealing interpretation from a combinatorial point of view is that we have revealed a circulation of the particles, that use the bottom row to travel backward and implement the infinite reservoirs, as illustrated by Figure 4. An example of evolution is given by Figure 8. The TASEP and complete chain with two particles for $n = 3$ are represented in Figures 11–12.

2.3. Restatement of the transition rules: the bijection $\bar{T}$.

Theorem 2.3. *The application T is the first component $\Omega^0_n \times \{0, \ldots, n\} \to \Omega^0_n$ of a bijection $\bar{T}$ from $\Omega^0_n \times \{0, \ldots, n\}$ into itself.*

Proof. In order to define the application $\bar{T}$, we shall partition the set $\Omega^0_n \times \{0, \ldots, n\}$ into classes $A_{a'}$, $A_{a''}$, A_b, A_c, and A_d, and describe, for each class A, its image $B = \bar{T}(A)$. From now on in this section, (ω, i) and (ω', j) respectively denote an element of the current class and its image, and j_1 and j_2 are defined from (ω, i) as in Section 2.2.

We are going to describe the image (ω', j) of (ω, i) by $\bar{T}$ in terms of deletions and insertions of $|{}^{\bullet}_{\circ}|$- or $|{}^{\circ}_{\bullet}|$-columns or of ${}^{\bullet}|_{\circ}$-diagonals. One advantage of these operations is that they clearly preserve the balance and positivity conditions, so we will directly know in each case that the image ω' belongs Ω^0_n. The reader is invited to chek, using Figures 9 and 10, that the configuration ω' obtained in each case is, as claimed in the theorem, the same as the configuration $T(\omega, i)$ that was described in terms of sweeps in the previous section:

- If the wall i separates in the top row of ω a black particle P and a white particle Q. There are two cases depending on the type of the particle R that is below Q in ω:

  $A_{a'}$ The particle R is black. Then $j = j_1$ and ω' is obtained by moving the $|{}^{\circ}_{\bullet}|$-column $|{}^{Q}_{R}|$ from the right-hand side of wall i to the right-hand side of wall j (Figure 9, left-middle).

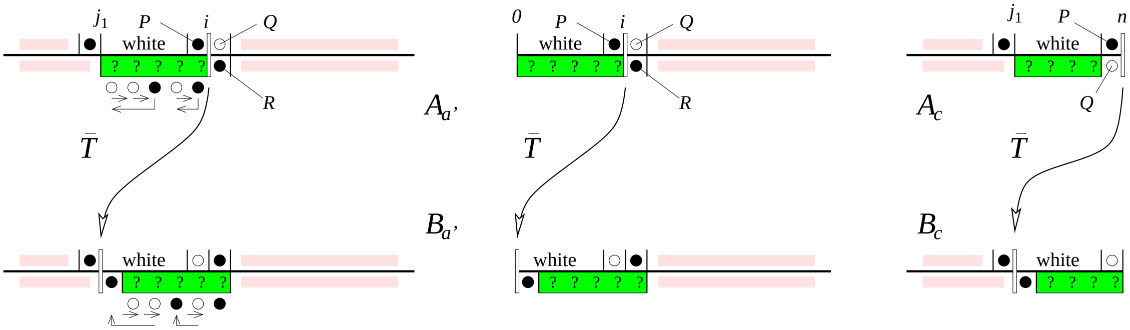

FIGURE 9. Moves in cases $A_{a'}$ and A_c. Below the two left-hand side configurations, the white sweep in the bottom row is illustrated on an exemple.

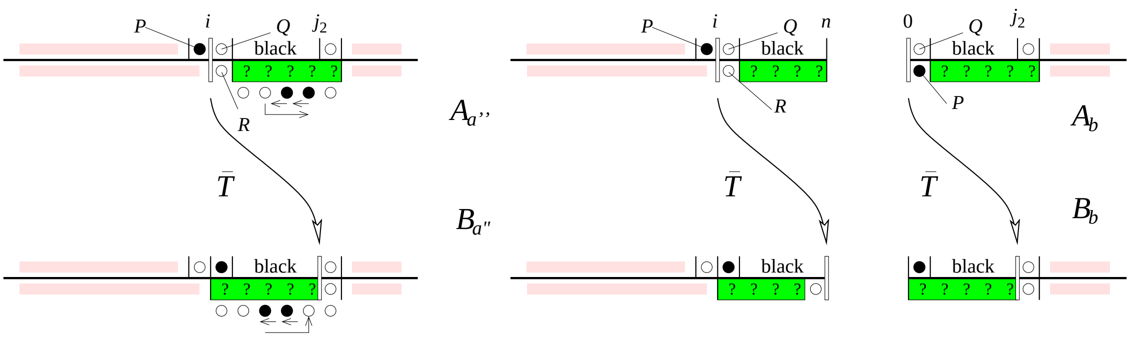

FIGURE 10. Moves in the cases A''_a and A_b. Below the two left-hand side configurations, the black sweep in the bottom row is illustrated on an exemple.

The image $B_{a'}$ of the class $A_{a'}$ consists of pairs (ω', j) such that: the wall j is the left border ($j = 0$) or it has a black particle on its left-hand side in the top row, there is a $|\substack{\circ\\ \bullet}|$-column on the right-hand side of wall j, and the sequence of white particles on the right-hand side of wall j in the top row is followed by a black particle.

$A_{a''}$ The particle R is white. Then $j = j_2$ and ω' is obtained by moving the particles P and R from wall i (where they form a $^\bullet|_\circ$-diagonal) to wall j so that they form a $^\bullet|_\circ$-diagonal if $j < n$ (Figure 10, left), or a $|\substack{\bullet\\ \circ}|$-column if $j = n$ (Figure 10, middle).
The image $B_{a''}$ of the class $A_{a''}$ consists of pairs (ω', j) with a $|\substack{\circ\\ \circ}|$-column or the border on the right-hand side of wall j of ω' and such that there is a non-empty sequence of black particles on the left-hand side of wall j in the top row, followed by a white particle.

A_b If $i = 0$ and there is a white particle Q in the leftmost top cell of ω. The cell under Q then contains a black particle P (Figure 10, right). Then $j = j_2$ and ω' is obtained by moving P and Q to wall j so that they form a $^\bullet|_\circ$-diagonal if $j < n$ or a $|\substack{\bullet\\ \circ}|$-column if $j = n$.

The image B_b of the class A_b consists of pairs (ω', j) with a $|\substack{\circ\\ \circ}|$-column or the border on the right-hand side of wall j of ω' and such that there is a non-empty sequence of black particles on the left-hand side of wall j in the top row, ending at the left border.

A_c If $i = n$ and there is a black particle P in the rightmost top cell of ω. The cell under P then contains a white particle Q (Figure 9, right). Then $j = j_1$ and ω is obtained by moving P and Q to wall j so that they form a $|\substack{\circ\\ \bullet}|$-column on its right-hand side.

The image B_c of the class A_c consists of pairs (ω', j) such that: the wall j is the left border ($j = 0$) or it has a black particle on its left-hand side in the top row, there is a $|\substack{\circ\\ \bullet}|$-column on the right-hand side of wall j, and the sequence of white particles on the right-hand side of wall j in the top row ends at the right border.

A_d This class contains all the remaining pairs (ω, i). These configurations are left unchanged by $\bar{T}$, so that $B_d = \bar{T}(A_d) = A_d$.

In each case of the definition, the transformation described is reversible: from (ω', k) in one of the image classes, the wall i and then the configuration ω are easily recovered. The theorem thus follows from the fact that $\{B_{a'}, B_{a''}, B_b, B_c, B_d\}$ is a partition of $\Omega_n^0 \times \{0, \ldots, n\}$. $\square$

3. Stationary distributions

The Markov chain $X^0_{\alpha\beta\gamma}$ is clearly aperiodic and we check in Section 6 that it is irreducible, *i.e.* that there is an evolution between any two configurations. This implies that the chain $X^0_{\alpha\beta\gamma}$ is ergodic, *i.e.* it has a unique stationary distribution, to which $X^0_{\alpha\beta\gamma}(t)$ converges as t goes to infinity [7]. Our aim in this section is to find this distribution and to use it to give a combinatorial interpretation to that of $S^0_{\alpha\beta\gamma}$.

We first deal with the maximal flow regime, for which all ingredients are now ready. Then we discuss the generic case.

3.1. The maximal flow regime $\alpha = \beta = \gamma = 1$.

Theorem 3.1. *The Markov chain X^0 has a uniform stationary distribution.*

Proof. As illustrated by Figure 12, Theorem 2.3 says that the vertices of the transition graph of the chain have equal in- and out-degrees. Moreover the $n+1$ possible transitions from a configuration ω have equal probabilities to be chosen, since the active wall is chosen uniformly in $\{0, \ldots, n\}$. The uniform distribution on Ω_n^0 hence clearly satisfies the local stationarity equation at each configuration ω: assuming that at time t the distribution is uniform,

$$\mathrm{Prob}(X(t) = \omega) = \frac{1}{|\Omega_n^0|} \qquad \text{for all } \omega,$$

then at time $t+1$, it remains uniform, since

$$\mathrm{Prob}(X(t+1) = \omega') = \sum_{(\omega,i) \in T^{-1}(\omega')} \mathrm{Prob}(X(t) = \omega) \cdot \frac{1}{n+1} \;=\; \left|T^{-1}(\omega')\right| \cdot \frac{1}{|\Omega_n^0|} \cdot \frac{1}{n+1} \;=\; \frac{1}{|\Omega_n^0|},$$

where $T^{-1}(\omega')$ denotes the set of preimages of ω' respectively by T. The last equality follows from the facts that $T^{-1}(\omega') = \{\bar{T}^{-1}(\omega', j) \mid j = 0, \ldots, n\}$, and that $\bar{T}$ is a bijection. $\square$

The relation $S^0 \equiv \mathrm{top}(X^0)$ now allows us to derive from Theorem 3.1 the annonced combinatorial interpretation of Formulas (1.1) and (1.2).

Theorem 3.2. *Let $top(\omega)$ denote the top row of a complete configuration ω. Then for any initial distribution $S^0(0)$ and $X^0(0)$ with $S^0(0) \equiv top(X^0(0))$, and any TASEP configuration τ,*

$$Prob(S^0(t) = \tau) \;=\; Prob(top(X^0(t)) = \tau) \underset{t\to\infty}{\longrightarrow} \frac{\left|\{\omega \in \Omega_n^0 \mid top(\omega) = \tau\}\right|}{|\Omega_n^0|}.$$

In particular, for any $k + m = n$, we obtain combinatorially the formula:

$$Prob(S^0(t) \text{ contains } k \text{ black and } m \text{ white particles}) \underset{t\to\infty}{\longrightarrow} \frac{|\Omega^0_{k,m}|}{|\Omega_n^0|} \;=\; \frac{\frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{m}}{C_{n+1}}.$$

As discussed in Section 8, this interpretation sheds a new light on some recent results of Derrida *et al.* connecting the TASEP to Brownian excursions [4].

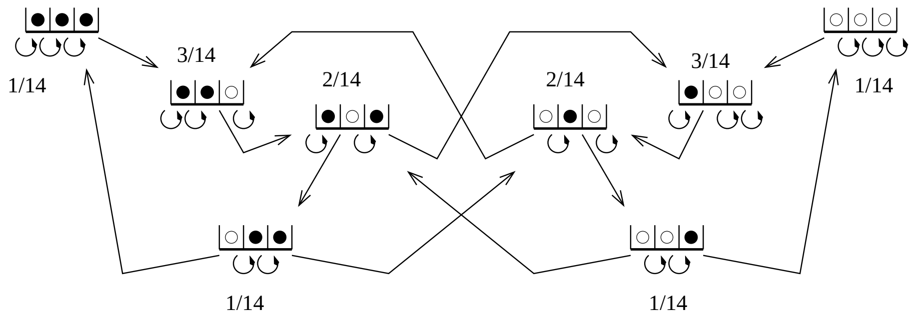

FIGURE 11. The TASEP configurations for $n = 3$ and transitions between them. The starting point of each arrow indicates the wall triggering the transition. The numbers are the stationary probabilities for $\alpha = \beta = \gamma = 1$.

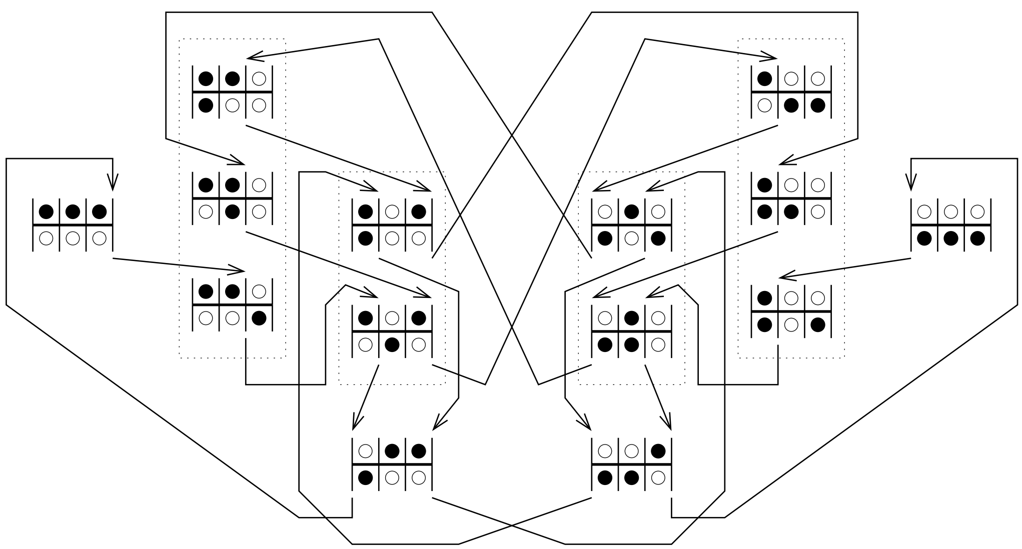

FIGURE 12. The 14 complete configurations for $n = 3$ and transitions between them. The starting point of each arrow indicates the wall triggering the transition (loop transitions are not indicated). For $\alpha = \beta = \gamma = 1$, the stationary probabilities are uniform (equal to 1/14) since each configuration has equal in- and out-degrees. Ignoring the bottom rows reduces this Markov chain to the chain of Figure 11.

3.2. **Arbitrary α, β and γ.** In order to express the stationary distribution of the general chain $X^0_{\alpha\beta\gamma}$, we associate a weight $q(\omega)$ to each complete configuration ω, which is defined in terms of two combinatorial statistics.

By definition, a complete configuration ω is a concatenation of four types of columns $|{\bullet \atop \bullet}|$, $|{\bullet \atop \circ}|$, $|{\circ \atop \bullet}|$ and $|{\circ \atop \circ}|$, subject to the balance and positivity conditions. In particular, the concatenation of two complete configurations of Ω^0_i and Ω^0_j with $i+j = n$ yields a complete configuration of Ω^0_n. Let us call *prime* a configuration that cannot be decomposed in this way. A complete configuration ω can be uniquely written as a concatenation $\omega = \omega_1 \cdots \omega_m$ of prime configurations. These prime factors can be of three types: $|{\bullet \atop \circ}|$-columns, $|{\circ \atop \bullet}|$-columns, and *blocks* of the form $|{\bullet \atop \bullet}|\omega'|{\circ \atop \circ}|$ with ω' a complete configuration. The inner part ω' of a block $\omega = |{\bullet \atop \bullet}|\omega'|{\circ \atop \circ}|$ is referred to as its *inside*.

Now, given a complete configuration ω, let us assign labels to some of the particles of its bottom row: first, each white particle is labeled z if it is not in a block, and then, each black particle is labeled y if it is not in the inside of a block and there are no z labels on its left. The number of

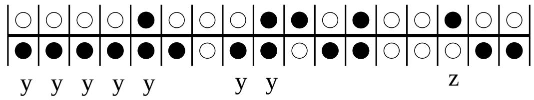

FIGURE 13. A configuration ω with weight $q(\omega) = \alpha^8\beta^{10}\gamma^{16}$. Labels are indicated below particles.

labels of type y and the number of labels of type z in a configuration ω will be denoted $n_y(\omega)$ and $n_z(\omega)$ respectively. Then the *weight* of a configuration ω is defined as

$$q(\omega) \;=\; \beta^n\gamma^n\left(\frac{\alpha}{\beta}\right)^{n_y(\omega)}\left(\frac{\alpha}{\gamma}\right)^{n_z(\omega)} \;=\; \alpha^{n_y(\omega)+n_z(\omega)}\beta^{n-n_y(\omega)}\gamma^{n-n_z(\omega)}.$$

In other terms, there is a factor α per label, a factor β per unlabeled black particle and a factor γ per unlabeled white particle. For instance, the weight of the configuration of Figure 13 is $\alpha^8\beta^{10}\gamma^{16}$, and more generally the weight is a monomial with total degree $2n$.

Theorem 3.3. *The Markov chain $X^0_{\alpha\beta\gamma}$ has the following unique stationary distribution:*

$$Prob(X^0_{\alpha\beta\gamma}(t) = \omega) \underset{t\to\infty}{\longrightarrow} \frac{q(\omega)}{Z^0_n}, \qquad \text{where } Z^0_n = \sum_{\omega'\in\Omega^0_n} q(\omega'),$$

where $q(\omega)$ is the previously defined weight on complete configurations.

Since $X^0_{\alpha\beta\gamma}$ is aperiodic and irreducible, it is sufficient to show that the distribution induced by the weights q is stationary. The result is based on a further property of the bijection $\bar{T}$ of which T is the first component.

Lemma 3.4. *The bijection $\bar{T} : \Omega^0_n \times \{0, \dots, n\} \to \Omega^0_n \times \{0, \dots, n\}$ transports the weights:*

$$\lambda(i)\, q(\omega) \;=\; \lambda(j)\, q(\omega'), \qquad \text{for all } (\omega', j) = \bar{T}(\omega, i), \tag{3.1}$$

where $\lambda(i) = \alpha$ for $i \in \{1, \dots, n-1\}$, $\lambda(0) = \beta$ and $\lambda(n) = \gamma$.

Proof. Let ω be a complete configuration belonging to Ω^0_n. The following properties will be useful:

- Property 1. *In a local configuration $|{}^{\bullet}_{?}|{}^{\circ}_{\bullet}|$ the black particle in the bottom row never contributes a label y.*

  The black particle of a $|{}^{\circ}_{\bullet}|$-column can contribute a label y only if it is not in the inside of a block. This happens only if the particle ? is white and is not in a block. But then this particle carries a label z which is on the left of the black particle.
- Property 2. *The bottom white particle of a $|{}^{\circ}_{\circ}|$-column never contributes a label z.*

  This property is immediate since a $|{}^{\circ}_{\circ}|$-column is always in a block.
- Property i. *The deletion/insertion of a $|{}^{\circ}_{\bullet}|$-column does not change the labels of other particles.*

  When a $|{}^{\circ}_{\bullet}|$-column is inserted or removed in the inside of a block, the block structure is unchanged and there is no effect on labels. When it is inserted or removed at a position no included in a block it may contribute a label y, but this has no effect on other labels.
- Property ii. *The deletion/insertion of a ${}^{\bullet}|_{\circ}$-diagonal taking the form $|{}^{\bullet}_{?}|{}^{\circ}_{\circ}| \leftrightarrow |{}^{\circ}_{?}|$ does not change the labels of other particles.*

  The situation $|{}^{\bullet}_{\circ}|{}^{\circ}_{\circ}| \leftrightarrow |{}^{\circ}_{\circ}|$ can be view as the insertion or deletion of a $|{}^{\bullet}_{\circ}|$-column in the inside of a block, which has no effect. The other situation $|{}^{\bullet}_{\bullet}|{}^{\circ}_{\circ}| \leftrightarrow |{}^{\circ}_{\bullet}|$ may occur outside a block, in which case a white particle is added or removed on the bottom row, but in a small block $|{}^{\bullet}_{\bullet}|{}^{\circ}_{\circ}|$.

The relation is checked using these properties by comparing $q(\omega)$ and $q(\omega')$ in each case of the definition of the bijection $\bar{T}$.

$A_{a'}$ If $j \neq 0$ (Figure 9, left), according to Property 1 the particle R does not contribute a label y neither in ω nor in ω'. Moreover, according to Property i, the displacement of the $|{}^{\circ}_{\bullet}|$-column does not affect labels of other particles. Hence $q(\omega) = q(\omega')$, in agreement with $\lambda(i) = \lambda(j) = \alpha$.

If instead $j = 0$ (Figure 9, middle), Property 1 applies only to ω: in ω', the displaced $|\substack{\circ \\ \bullet}|$-column is the leftmost one, so that its black particle contributes a supplementary y label. Therefore $q(\omega') = q(\omega)\frac{\alpha}{\beta}$, in agreement with $\lambda(i) = \alpha$, $\lambda(0) = \beta$.

$A_{a''}$ If $j \neq n$ (Figure 10, left), from Property 2 we see that the particle R does not contribute a label y neither in ω nor in ω'. Observe moreover that the displacement of a ${}^{\bullet}|_{\circ}$-diagonal does not affect labels of other particles according to Property ii. Hence $q(\omega) = q(\omega')$, in agreement with $\lambda(i) = \lambda(j) = \alpha$.

If $j = n$ (Figure 10, middle), Property 2 applies only to ω: the move amounts to deleting a ${}^{\bullet}|_{\circ}$-diagonal and inserting a $|\substack{\bullet \\ \circ}|$-column at the right border. The white particle of this column thus contributes a z label. Therefore $q(\omega') = q(\omega)\frac{\alpha}{\gamma}$, in agreement with $\lambda(i) = \alpha$ and $\lambda(n) = \gamma$.

A_b If $j \neq n$ (Figure 10, right), the move consists in deleting a $|\substack{\circ \\ \bullet}|$-column, which is the leftmost and thus contributes a y label in ω, and inserting a ${}^{\bullet}|_{\circ}$-diagonal, which according to Property 2 does not contribute a z label. According to Property i and ii the other labels are left unchanged. Therefore $q(\omega') = q(\omega)\frac{\beta}{\alpha}$, in agreement with $\lambda(0) = \beta$ and $\lambda(j) = \alpha$.

If $j = n$, $q(\omega') = q(\omega)\frac{\beta}{\alpha}\frac{\alpha}{\gamma} = q(\omega)\frac{\beta}{\gamma}$, in agreement with $\lambda(0) = \beta$ and $\lambda(n) = \gamma$.

A_c If $j \neq 0$ (Figure 9, right), ω' is obtained by deleting a $|\substack{\bullet \\ \circ}|$-column on the left-hand side of the wall n and inserting a $|\substack{\circ \\ \bullet}|$-column on the right-hand side of j_1. According to Property i only the labels of displaced particles can be affected. Since the deleted $|\substack{\bullet \\ \circ}|$-column is the rightmost column, its white particle contributes a z label in ω. As opposed to that, Property 1 forbids the $|\substack{\circ \\ \bullet}|$-column to contribute a label in ω'. Therefore $q(\omega') = q(\omega)\frac{\gamma}{\alpha}$, in agreement with $\lambda(n) = \gamma$ and $\lambda(j) = \alpha$.

If $j = 0$, $q(\omega') = q(\omega)\frac{\gamma}{\alpha}\frac{\alpha}{\beta} = q(\omega)\frac{\gamma}{\beta}$, in agreement with $\lambda(n) = \gamma$ and $\lambda(0) = \beta$.

□

Proof of Theorem 3.3. In order to see that the distribution induced by q is stationary, let us assume that

$$\mathrm{Prob}(X^0_{\alpha\beta\gamma}(t) = \omega) \ = \frac{q(\omega)}{Z^0_n}, \qquad \text{for all } \omega \in \Omega^0_n,$$

and try to compute $\mathrm{Prob}(X^0_{\alpha\beta\gamma}(t+1) = \omega')$. For this, recall that $I(t)$ denotes the random wall selected at time t and define $J(t+1)$ as follows: if $I(t)$ becomes active so that $X^0_{\alpha\beta\gamma}(t+1) = T(X^0_{\alpha\beta\gamma}(t), I(t))$, then define $J(t+1)$ by $\bar{T}(X^0_{\alpha\beta\gamma}(t), I(t)) = (X^0_{\alpha\beta\gamma}(t+1), J(t+1))$; otherwise set $J(t+1) = I(t)$. Then, since T is given as the first component of $\bar{T}$,

$$\mathrm{Prob}\big(X^0_{\alpha\beta\gamma}(t+1) = \omega'\big) \ = \ \sum_{j=0}^{n} \mathrm{Prob}\big(X^0_{\alpha\beta\gamma}(t+1) = \omega',\ J(t+1) = j\big).$$

Now, by definition of the Markov chain $X^0_{\alpha\beta\gamma}$, for all ω' and j,

$$\begin{aligned} \mathrm{Prob}\big(X^0_{\alpha\beta\gamma}(t+1) = \omega',\ J(t+1) = j\big) \ &= \ \lambda(i) \cdot \mathrm{Prob}\big(X^0_{\alpha\beta\gamma}(t) = \omega,\ I(t) = i\big) \\ &\quad + (1 - \lambda(j)) \cdot \mathrm{Prob}\big(X^0_{\alpha\beta\gamma}(t) = \omega',\ I(t) = j\big), \end{aligned}$$

where $(\omega, i) = \bar{T}^{-1}(\omega', j)$. Since the random variable $I(t)$ is uniform on $\{0, \ldots, n\}$, we get

$$\mathrm{Prob}\big(X^0_{\alpha\beta\gamma}(t+1) = \omega',\ J(t+1) = j\big) \ = \ \lambda(i) \cdot \frac{q(\omega)}{Z^0_n}\frac{1}{n+1} \ + \ (1 - \lambda(j)) \cdot \frac{q(\omega')}{Z^0_n}\frac{1}{n+1}.$$

But since $\bar{T}$ preserves the weights via Relation (3.1), $\lambda(i)q(\omega) = \lambda(j)q(\omega')$ and the terms involving λ cancel. Finally

$$\mathrm{Prob}(X^0_{\alpha\beta\gamma}(t+1) = \omega') \ = \ \sum_{j=0}^{n} \frac{q(\omega')}{Z^0_n}\frac{1}{n+1} \ = \ \frac{q(\omega')}{Z^0_n},$$

and this completes the proof that the distribution induced by q is stationary. □

4. The 3-TASEP

The combinatorial approach we developed in the previous sections can be extended to a slightly more general model, the 3-TASEP, which we now define. The 3-TASEP is similar to the TASEP but each time a black or a white particle exits, there is a certain probability ε that the particle that enter in its place is a neutral particle $\times$. On the one hand, as in the TASEP, black particles always travel from left to right and white particles always do the opposite. On the other hand, neutral particles have no preferred direction and get displaced in opposite direction by black and white particles. An informal illustration of the 3-TASEP is given by Figure 14.

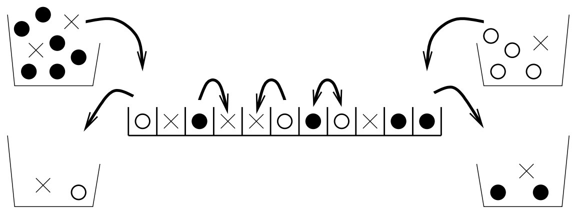

Figure 14. The 3-TASEP.

4.1. **Definition of the 3-TASEP.**

A *3-TASEP configuration* is a row of n cells, each containing one particle, which can be of type $\bullet$, $\times$ or $\circ$. An example of configuration is given by Figure 15. The local configuration around a wall i in a configuration τ is denoted $\tau[i]$: for $i \in \{1, \dots, n-1\}$, $\tau[i]$ is the element of the set $\{\bullet\|\times, \bullet\|\circ, \bullet\|\bullet, \times\|\bullet, \times\|\times, \times\|\circ, \circ\|\bullet, \circ\|\times, \circ\|\circ\}$ that describes the two cells surrounding wall i, for $i = 0$, $\tau[0] \in \{\|\bullet, \|\times, \|\circ\}$, and for $i = n$, $\tau[n] \in \{\bullet\|, \times\|, \circ\|\}$.

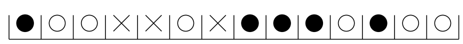

Figure 15. A 3-TASEP configuration with $n = 14$ cells.

The 3-TASEP is a Markov chain $S_{\alpha\beta\gamma\varepsilon}$ defined on the set of 3-TASEP configurations in terms of four parameters α, β and γ in $]0, 1]$ and ε in $[0, 1]$. From time t to $t+1$, the chain evolves from the configuration $\tau = S_{\alpha\beta\gamma\varepsilon}(t)$ to a configuration $\tau' = S_{\alpha\beta\gamma\varepsilon}(t+1)$ as follows:

- A wall i is chosen uniformly at random among the $n+1$ walls.
- Depending on the local configuration $\tau[i]$ around wall i, a transition may be triggered:
  - unstable local configurations in the middle ($i \in \{1, \dots, n-1\}$):
    - a_1. Case $\bullet\|\circ$, a transition $\bullet\|\circ \to \circ|\bullet$ occurs with probability $\lambda(\bullet\|\circ) := \alpha$.
    - a_2. Case $\times\|\circ$, a transition $\times\|\circ \to \circ|\times$ occurs with probability $\lambda(\times\|\circ) := \beta$.
    - a_3. Case $\bullet\|\times$, a transition $\bullet\|\times \to \times|\bullet$ occurs with probability $\lambda(\bullet\|\times) := \gamma$.
  - unstable local configurations on the left border ($i = 0$):
    - b_1. Case $\|\circ$, the particle exits with total probability $\lambda(\|\circ) := \beta$, in 2 possible ways:
      - b_1'. a transition $\|\circ \to |\bullet$ occurs with probability $(1-\varepsilon)\,\beta$,
      - b_1''. or a transition $\|\circ \to |\times$ with probability $\varepsilon\beta$ (neutralization),
    - b_2. Case $\|\times$, a transition $\|\times \to |\bullet$ occurs with probability $\lambda(\|\times) := (1-\varepsilon)\beta\,\gamma/\alpha$.
  - unstable local configurations on the right border ($i = n$):
    - c_1. Case $\bullet\|$, the particle exits with total probability $\lambda(\bullet\|) := \gamma$, in 2 possible ways:
      - c_1'. a transition $\bullet\| \to \circ|$ occurs with probability $(1-\varepsilon)\,\gamma$,
      - c_1''. or a transition $\bullet\| \to \times|$ with probability $\varepsilon\gamma$ (neutralization),
    - c_2. Case $\times\|$, a transition $\times\| \to \circ|$ occurs with probability $\lambda(\times\|) := (1-\varepsilon)\gamma\,\beta/\alpha$.
  - stable local configurations:
    - *d*. Cases $\bullet\|\bullet$, $\times\|\times$, $\circ\|\circ$, $\circ\|\times$, $\circ\|\bullet$, $\times\|\bullet$, $\|\bullet$ and $\circ\|$, no transition occur: $\lambda(\bullet\|\bullet) = \lambda(\times\|\times) = \lambda(\circ\|\circ) = \lambda(\circ\|\times) = \lambda(\circ\|\bullet) = \lambda(\times\|\bullet) = \lambda(\|\bullet) = \lambda(\circ\|) := 0$.
- If a transition occurs, the new configuration τ' is obtained from τ by applying the transition to the local configuration around the chosen wall. Otherwise, $\tau' = \tau$.

In order to explain the role of the parameters α, β, γ and ε a few remarks are useful:

- The equality $\lambda(\times\|\circ) = \lambda(\|\circ)$ translates the idea that a white particle feels the same attraction to the left in front of a neutral particle as it feels for exiting at the left border. A similar interpretation holds for $\lambda(\bullet\|\times) = \lambda(\bullet\|)$ and black particles.
- The equality $\lambda(\times\|)/\lambda(\times\|\circ) = (1-\varepsilon)\lambda(\bullet\|)/\lambda(\bullet\|\circ)$, or says that the ratio between entry and movement rates for white particles is the same in presence of black or neutral particles. A similar interpretation holds for black particles.
- The fact that the same quantity ε controls the probability that a $\times$ particles enters instead of a black particle or instead of a white particle may be thought of as a curious restriction: it is dictated by technical considerations in the proof, and at the present state we do not know whether it can be easily circumvented or not.

The TASEP with parameter α, β and γ is recovered by taking $\varepsilon = 0$. Indeed, in this case, after the initial neutral particles have exit the system, no new neutral particles are created and the rules are exactly those of the TASEP as presented in Section 1.

It will be useful to reformulate again the transition of the 3-TASEP in terms of applications from the set of configurations with a chosen wall into the set of configurations. Since there are two possible transitions in the cases $\|\circ$ and $\bullet\|$ we introduce the following two applications:

- The application $\vartheta_1 : (\tau, i) \to \tau'$ performing at wall i the transitions prescribed by cases a_1, a_2 and a_3, b_1' and b_2, c_1' and c_2.
- The application $\vartheta_2 : (\tau, i) \to \tau'$ performing at wall i the transitions prescribed by cases a_1, a_2 and a_3, b_1'' and b_2, c_1'' and c_2.

Then the transitions of the chain $S_{\alpha\beta\gamma\varepsilon}$ can be described as follows: choose $i = I(t)$ uniformly at random in $\{0, \dots, n\}$ and set

$$S_{\alpha\beta\gamma\varepsilon}(t+1) = \begin{cases} \vartheta_1(\tau, i) & \text{with probability} \quad (1-\varepsilon)\lambda(\tau[i]), \\ \vartheta_2(\tau, i) & \text{with probability} \quad \varepsilon\lambda(\tau[i]), \\ \tau & \text{otherwise,} \end{cases}$$

where $\tau = S_{\alpha\beta\gamma\varepsilon}(t)$, and $\tau[i]$ denotes the local configuration around wall i in τ.

4.2. **Complete configurations for the 3-TASEP.** The complete configurations for the 3-TASEP are concatenations of complete configurations for the TASEP separated by $|^{\times}_{\times}|$-columns: more explicitly, each complete configuration ω with ℓ $\times$-particles in the first row can be uniquely written $\omega_0|^{\times}_{\times}|\omega_1 \cdots |^{\times}_{\times}|\omega_\ell$ where each ω_i belongs to Ω_{n_i} for some $n_i \geq 0$. In other terms, these complete configurations are pairs of rows of cells containing particles such that the $\times$-particles always form $|^{\times}_{\times}|$-columns and such that between two $|^{\times}_{\times}|$-columns the balance and positivity conditions are satisfied. Let Ω_n denote the set of complete configurations of length n.

●	○	○	×	×	○	×	●	●	●	○	●	○	○
●	●	○	×	×	●	×	○	●	○	●	●	○	○

FIGURE 16. A complete configuration with $n = 14$.

An example of a complete configuration is given in Figure 16: from left to right the subconfigurations have successively length 3, 0, 1, and 7. The local configuration around wall i in a complete configuration ω, describing the one or two columns surrounding wall i, is denoted $\omega[i]$. The following enumerative Lemmas are proved in Section 7.

Lemma 4.1. *The cardinality of* Ω_n *is* $\frac{1}{2}\binom{2n+2}{n+1}$.

Lemma 4.2. *For any* $k + \ell + m = n$, *the cardinality of the set* $\Omega^{\ell}_{k,m}$ *of complete configurations of* Ω_n *with* ℓ $|^{\times}_{\times}|$*-columns, and* k *black and* m *white particles on the top row is* $\frac{\ell+1}{n+1}\binom{n+1}{k}\binom{n+1}{m}$.

Lemma 4.3. *For any* $\ell + p = n$, *the cardinality of the set* Ω^{ℓ}_{n} *of complete configurations of* Ω_n *with* ℓ $|^{\times}_{\times}|$*-columns is* $\frac{\ell+1}{n+1}\binom{2n+2}{n-\ell}$.

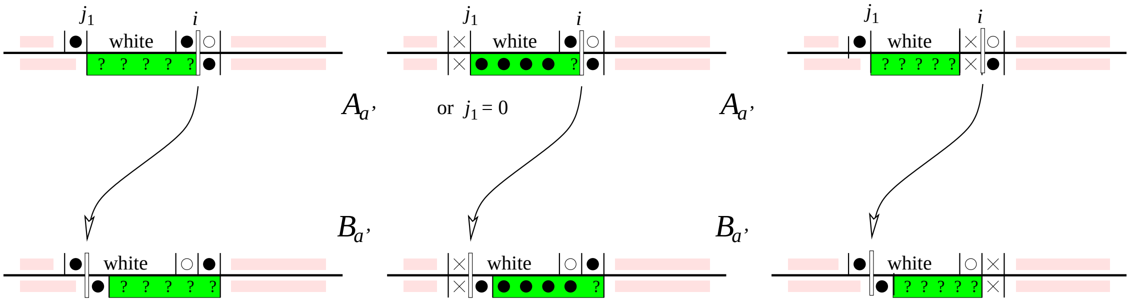

FIGURE 17. Cases $\bullet\|\circ$ and $\times\|\circ$ for the bijections $\bar{T}_1$ and $\bar{T}_2$.

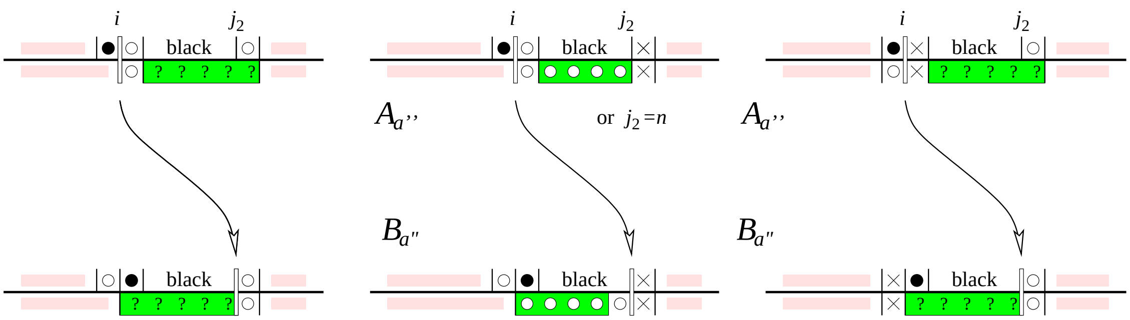

FIGURE 18. Cases $\bullet\|\circ$ and $\bullet\|\times$ with black sweep for the bijections $\bar{T}_1$ and $\bar{T}_2$.

4.3. **The complete chain $X_{\alpha\beta\gamma\varepsilon}$.** We shall directly define the chain $X_{\alpha\beta\gamma\varepsilon}$ in terms of two bijections $\bar{T}_1$ and $\bar{T}_2$ from $\Omega_n \times \{0, \ldots, n\}$ to itself. To do this, we partition the set $\Omega_n \times \{0, \ldots, n\}$ into classes, and we first describe for each class A the image (ω', j) of a pair $(\omega, i) \in A$ by $\bar{T}_1$. The bijection $\bar{T}_2$ is then described as a simple variation on $\bar{T}_1$.

As in Section 2.2, given a complete configuration ω with top row τ and a wall i, we distinguish the following walls: if the local configuration $\tau[i]$ is $\bullet\|\circ$, $\times\|\circ$, $\bullet\|$ or $\times\|$, then let $j_1 < i$ be the leftmost wall such that there are only white particles in the top row between walls j_1 and $i-1$; if the local configuration $\tau[i]$ is $\bullet\|\circ$, $\bullet\|\times$, $\|\circ$ or $\|\times$, then let $j_2 > i$ be the rightmost wall such that there are only black particles in the top row between walls $i+1$ and j_2.

The action of $\bar{T}_1$ is described separately for the different cases of local configuration $\omega[i]$:

- unstable local configurations in the middle ($i \in \{1, \ldots, n-1\}$):

  $A_{a'}$. Cases ${}^{\bullet}_{?}\|^{\circ}_{\bullet}$ and ${}^{\times}_{\times}\|^{\circ}_{\bullet}$. Then $j = j_1$, and ω' is obtained by moving the $|^{\circ}_{\bullet}|$-column from the right-hand side of wall i to the right-hand side of wall j (Figure 17).
  The image $B_{a'}$ of this class consists of pairs (ω', j) such that: the wall j is the left border ($j = 0$) or there is a black or a $\times$ particle on its left-hand side, there is a $|^{\circ}_{\bullet}|$-column on its right-hand side, and the sequence of white particles on the right-hand side of wall j in the top row is followed by a black or a $\times$ particle.

  $A_{a''}$. Cases ${}^{\bullet}_{?}\|^{\circ}_{\circ}$ or ${}^{\bullet}_{\circ}\|^{\times}_{\times}$. Then $j = j_2$, and ω' is obtained by removing the two particles that form the ${}^{\bullet}|_{\circ}$-diagonal or the $|^{\bullet}_{\circ}|$-column at wall i and replacing them at wall j so that they form a ${}^{\bullet}|_{\circ}$-diagonal if there is a white particle on the right-hand side of wall j in the top row, or a $|^{\bullet}_{\circ}|$-column otherwise (Figure 18).
  The image $B_{a''}$ of this class consists of pairs (ω', j) with a $|^{\circ}_{\circ}|$-column, an $|^{\times}_{\times}|$-column, or the border on the right-hand side of wall j and such that there is a non-empty sequence of black particles on the left-hand side of wall j in the top row, followed by a white or an $\times$ particle.

- unstable local configurations on the left border ($i = 0$):

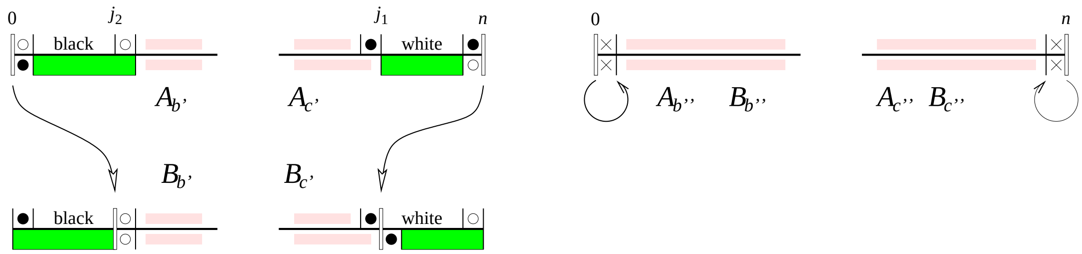

FIGURE 19. The application $\bar{T}_1$ on the borders.

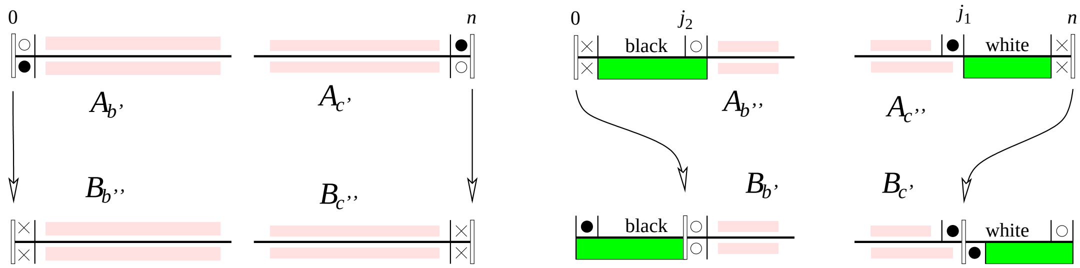

FIGURE 20. The application $\bar{T}_2$ on the borders.

$A_{b'}$. Case $\|\substack{\circ\\ \bullet}$. Then $j = j_2$, and ω' is obtained by removing the two particles that form the $|\substack{\circ\\ \bullet}|$-column on the left border and replacing them at wall j so that they form a ${}^{\bullet}|_{\circ}$-diagonal if there is a white particle on the right-hand side of wall j in the top row, or a $|\substack{\bullet\\ \circ}|$-column otherwise.
The image $B_{b'}$ of the class $A_{b'}$ by $\bar{T}_1$ consists of pairs (ω', j) with a $|\substack{\circ\\ \circ}|$-column, an $|\substack{\times\\ \times}|$-column, or the border on the right-hand side of wall j of ω' and such that there is a non-empty sequence of black particles on the left-hand side of wall j in the top row, ending at the left border.

$A_{b''}$. Case $\|\substack{\times\\ \times}$. Then $\omega' = \omega$ and $j = 0$. The image $B_{b''}$ of the class $A_{b''}$ by $\bar{T}_1$ consists of pairs $(\omega', 0)$ with a $|\substack{\times\\ \times}|$-column on the left border.

- unstable local configurations on the right border ($i = n$):

$A_{c'}$ Case $\substack{\bullet\\ \circ}\|$. Then $j = j_1$, and ω' is obtained by removing the rightmost $|\substack{\bullet\\ \circ}|$-column and forming a $|\substack{\circ\\ \bullet}|$-column on the right-hand side of wall j.
The image $B_{c'}$ of the class $A_{c'}$ by $\bar{T}_1$ consists of pairs (ω', j) such that: the wall j is the left border ($j = 0$) or it has a black or a $\times$ particle on its left-hand side, there is a $|\substack{\circ\\ \bullet}|$-column on its right-hand side, and the sequence of white particles on the right-hand side of wall j in the top row ends at the right border.

$A_{c''}$. Case $\substack{\times\\ \times}\|$. Then $\omega' = \omega$ and $j = n$. The image $B_{c''}$ of the class $A_{c''}$ by $\bar{T}_1$ consists of pairs (ω', n) with a $|\substack{\times\\ \times}|$-column on the right border.

Finally let A_d denote the set of pairs (ω, i) such that the local configuration around wall i of ω is stable. The mapping $\bar{T}_1$ has no effect on these pairs, and $B_d = A_d$.

The application is invertible in each case and the sets $\{B_{a'}, B_{a''}, B_{b'}, B_{b''}, B_{c'}, B_{c''}, B_d\}$ form a partition of $\Omega_n \times \{0, \ldots, n\}$. Hence $\bar{T}_1$ is a bijection from $\Omega_n \times \{0, \ldots, n\}$ onto itself.

The application $\bar{T}_2$ differs from $\bar{T}_1$ only at the borders. Consider the involution Y on $\Omega_n \times \{0, \ldots, n\}$ that acts only on a pair (ω, i) by changing, if $i = 0$ or $i = n$, the local configuration $\omega[i]$ according to the following rules:

$$\|\substack{\circ\\ \bullet} \leftrightarrow \|\substack{\times\\ \times}, \qquad \text{and} \qquad \substack{\bullet\\ \circ}\| \leftrightarrow \substack{\times\\ \times}\|.$$

Then the image of (ω, i) by $\bar{T}_2$ is defined to be the image of $Y(\omega, i)$ by $\bar{T}_1$. In particular $\bar{T}_2$, being the composition $\bar{T}_1 \circ Y$ of two bijections is itself a bijection. The action of $\bar{T}_2$ on the borders is illustrated by Figure 20.

Let now T_1 and T_2 denote respectively the first component of $\bar{T}_1$ and $\bar{T}_2$. Then the Markov chain $X_{\alpha\beta\gamma\varepsilon}$ is defined in terms of the T_1 and T_2 exactly as $S_{\alpha\beta\gamma\varepsilon}$ is defined in terms of the ϑ_1 and ϑ_2: choose $i = I(t)$ uniformly at random in $\{0, \ldots, n\}$ and set

$$X_{\alpha\beta\gamma\varepsilon}(t+1) = \begin{cases} T_1(\omega, i) & \text{with probability} \quad (1-\varepsilon)\lambda(\tau[i]), \\ T_2(\omega, i) & \text{with probability} \quad \varepsilon\lambda(\tau[i]), \\ \omega & \text{otherwise,} \end{cases}$$

where $\omega = X_{\alpha\beta\gamma\varepsilon}(t)$, and $\tau = \mathrm{top}(\omega)$.

4.4. **The stationary distribution of $X_{\alpha\beta\gamma\varepsilon}$ and $S_{\alpha\beta\gamma\varepsilon}$.** The parameter n_y and n_z of Section 3.2 are extended in a straightforward way to complete configurations of Ω_n by putting labels independently in each subconfiguration delimited by $|{}^{\times}_{\times}|$-columns or borders. Then for $\omega \in \Omega_n$, set

$$q(\omega) \;=\; \beta^n \gamma^n (1-\varepsilon)^n \left(\frac{\alpha}{\beta}\right)^{n_y(\omega)} \left(\frac{\alpha}{\gamma}\right)^{n_z(\omega)} \left(\frac{\alpha^2\varepsilon}{\beta\gamma(1-\varepsilon)}\right)^{\ell(\omega)}.$$

where $\ell(\omega)$ denotes the number of $|{}^{\times}_{\times}|$-columns in ω.

Then Theorem 3.3 extends verbatim:

Theorem 4.4. *The Markov chain $X_{\alpha\beta\gamma\varepsilon}$ has the following unique stationary distribution:*

$$Prob(X_{\alpha\beta\gamma\varepsilon}(t) = \omega) \underset{t\to\infty}{\longrightarrow} \frac{q(\omega)}{Z_n}, \qquad \text{where } Z_n = \sum_{\omega' \in \Omega_n} q(\omega'),$$

where $q(\omega)$ is the previously defined weight on the complete configurations of the 3-TASEP.

Again this theorem immediately yields a combinatorial interpretation of the stationary distribution of the chain $S_{\alpha\beta\gamma\varepsilon}$, via the relation $S_{\alpha\beta\gamma\varepsilon} = \mathrm{top}(X_{\alpha\beta\gamma\varepsilon})$. In particular in the case $\alpha = \beta = \gamma = 1$, $\varepsilon = 1/2$, we obtain the following corollary on $S = S_{111\frac{1}{2}}$ and $X = X_{111\frac{1}{2}}$:

Corollary 4.5. *Let $top(\omega)$ denote the top row of a complete configuration ω. Then for any initial distributions $S(0)$ and $X(0)$ with $top(X(0)) = S(0)$, and any basic configuration τ,*

$$Prob(S(t) = \tau) \;=\; Prob(top(X(t)) = \tau) \underset{t\to\infty}{\longrightarrow} \frac{\big|\{\omega \in \Omega_n \mid top(\omega) = \tau\}\big|}{|\Omega_n|}.$$

In particular, for any $k + \ell + m = n$, we obtain combinatorially the formula:

$$Prob(S(t) \text{ contains } k \text{ black and } m \text{ white particles}) \underset{t\to\infty}{\longrightarrow} \frac{|\Omega^{\ell}_{k,m}|}{|\Omega_n|} \;=\; \frac{\frac{\ell+1}{n+1}\binom{n+1}{k}\binom{n+1}{m}}{\frac{1}{2}\binom{2n+2}{n+1}}.$$

Theorem 4.4 is an easy consequence of the fact that the two bijections preserve weights in the sense of the following lemma. Recall that λ describes the transition probabilities for each possible local configuration.

Lemma 4.6. *The applications $\bar{T}_1$ and $\bar{T}_2$ transport together the weight λ in the following sense: for all $(\omega', j) \in \Omega \times \{0, \ldots, n\}$,*

$$(1-\varepsilon)\lambda(\omega_1[i_1])\, q(\omega_1) + \varepsilon\lambda(\omega_2[i_2])\, q(\omega_2) \;=\; \lambda(\omega'[j])\, q(\omega'),$$

where $(\omega_1, i_1) = \bar{T}_1^{-1}(\omega', j)$ and $(\omega_2, i_2) = \bar{T}_2^{-1}(\omega', j)$.

Proof. This lemma is easily verify by a case by case analysis similar to that of Lemma 3.4. ☐

Proof of Theorem 4.4. This proof exactly mimics the proof of Theorem 3.3, using Lemma 4.6 instead of Lemma 3.4. ☐

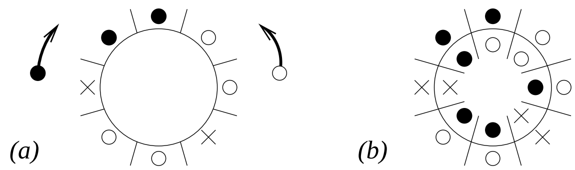

FIGURE 21. A basic and a complete configuration of the 3-TASEP with periodic boundary conditions.

5. Periodic boundary conditions

A standard alternative to our definition of the TASEP is to consider periodic boundary conditions: the leftmost cell is considered on the right-hand side of the rightmost cell, or equivalently, the configurations are arranged on a circle (see Figure 21 a., the circle is rigid, not subject to rotation).

Since there are no border walls in these configurations, the Markov chain $\widehat{S}_{\alpha\beta\gamma}$ is defined using only Cases a_1, a_2, a_3 of the transition of the 3-TASEP. Observer that the numbers k, ℓ and m of black, $\times$ and white particles do not change during the evolution. The case without $\times$ particle is easily seen to have a uniform stationary distribution, so we concentrate on the case with at least one $\times$ particle.

Our approach is easily adapted to deal with this case. Let $\widehat{\Omega}_n$ be a new set of complete configurations that are made of two rows of cells arranged on a circle and that are such that the subconfigurations between two $|^{\times}_{\times}|$-columns, when read in clockwise direction, satisfy the balance and positivity constraints. More precisely we are interested in the subset $\widehat{\Omega}^{\ell}_{k,m}$ of configurations of $\widehat{\Omega}_n$ that have ℓ $|^{\times}_{\times}|$-columns, k black and m white particles in the top row. The following lemma is proved in Section 7.

Lemma 5.1. *The cardinality of* $\widehat{\Omega}^{\ell}_{k,m}$ *is* $\binom{n}{k}\binom{n}{m}$.

Cases $A_{a'}$ and $A_{a''}$ of the definition of $\bar{T}_1$ allow to define a bijection $\widehat{T}$ from $\widehat{\Omega}^{\ell}_{k,m} \times \{0, \dots, n-1\}$ to itself and an associated Markov chain $\widehat{X}_{\alpha\beta\gamma}$ such that $\widehat{S}_{\alpha\beta\gamma} \equiv \mathrm{top}(\widehat{X}_{\alpha\beta\gamma})$. The same argument as in Section 3.1 for the chain X^0 then immediately yields the fact that $\widehat{X} = \widehat{X}_{111}$ has a uniform stationary distribution. In particular:

$$\mathrm{Prob}(\widehat{X}(t) = \omega) \underset{t\to\infty}{\longrightarrow} \frac{1}{|\widehat{\Omega}^{\ell}_{k,m}|} = \frac{1}{\binom{n}{k}\binom{n}{m}}.$$

Furthermore, the statistics n_y and n_z are immediately extended to configurations of $\widehat{\Omega}_n$ by putting label independently on every subconfiguration between $|^{\times}_{\times}|$-columns. Lemma 4.6 adapts in a straightforward way (with $\varepsilon = 0$), and allows to express the stationary distribution in the general case:

Theorem 5.2. *The Markov chain* $\widehat{X}_{\alpha\beta\gamma}$ *has the following unique stationary distribution:*

$$\mathit{Prob}(\widehat{X}_{\alpha\beta\gamma}(t) = \omega) \underset{t\to\infty}{\longrightarrow} \frac{q(\omega)}{\widehat{Z}_n}, \qquad \textit{where } \widehat{Z}_n = \sum_{\omega' \in \widehat{\Omega}_n} q(\omega'),$$

where $q(\omega) = \beta^n \gamma^n (\alpha/\beta)^{n_y(\omega)} (\alpha/\gamma)^{n_z(\omega)}$.

Finally the stationary distribution of $S_{\alpha\beta\gamma}$ is recovered from the relation $\widehat{S}_{\alpha\beta\gamma} \equiv \widehat{X}_{\alpha\beta\gamma}$. In particular, for $\widehat{S} = \widehat{S}_{111}$,

$$\mathrm{Prob}(\widehat{S}(t) = \tau) \underset{t\to\infty}{\longrightarrow} \frac{|\{\omega \in \widehat{\Omega}^{\ell}_{k,m} \mid \mathrm{top}(\omega) = \tau\}|}{|\widehat{\Omega}^{\ell}_{k,m}|}.$$

For instance, a configuration τ of the form

$$|\times|\underbrace{\circ\cdots\circ}_{m_1}|\underbrace{\bullet\cdots\bullet}_{k_1}|\times|\underbrace{\circ\cdots\circ}_{m_2}|\underbrace{\bullet\cdots\bullet}_{k_2}|\times|\cdots|\times|\underbrace{\circ\cdots\circ}_{m_\ell}|\underbrace{\bullet\cdots\bullet}_{k_\ell}|$$

for some $k_1+\ldots+k_\ell=k$, $m_1+\ldots+m_\ell=m$ corresponds to only one complete configuration

$$|{\times\atop\times}|\underbrace{{\circ\atop\bullet}\cdots{\circ\atop\bullet}}_{m_1}|\underbrace{{\bullet\atop\circ}\cdots{\bullet\atop\circ}}_{k_1}|{\times\atop\times}|\underbrace{{\circ\atop\bullet}\cdots{\circ\atop\bullet}}_{m_2}|\underbrace{{\bullet\atop\circ}\cdots{\bullet\atop\circ}}_{k_2}|{\times\atop\times}|\cdots|{\times\atop\times}|\underbrace{{\circ\atop\bullet}\cdots{\circ\atop\bullet}}_{m_\ell}|\underbrace{{\bullet\atop\circ}\cdots{\bullet\atop\circ}}_{k_\ell}|$$

(because of the positivity constraints on blocks between $|{\times\atop\times}|$-columns), and thus has probability $1/\binom{n}{k}\binom{n}{m}$ in the stationary distribution of $\widehat{S}$.

6. Irreducibility

In this section we verify that the Markov chains X^0, $\widehat{X}$ and X are irreducible, *i.e.* that there is a positive probability to go from any configuration ω to any other one ω'. In other terms we need to prove that the transition graphs of these three chains are connected. The proof is based on an observation about iterating the bijections $\bar{T}$, or $\bar{T}_1$ or $\bar{T}_2$, and on induction on n.

To every pair (ω,i) of $\Omega_n\times\{0,\ldots,n\}$ we associate a reduced configuration ω^i in Ω_{n-1}, obtained from ω by deleting two particles around the wall i in a way that depends on the local configuration:

- Cases ${\bullet\atop ?}\|{\circ\atop\bullet}$, ${\times\atop\times}\|{\circ\atop\bullet}$ and $\|{\circ\atop\bullet}$. The reduced configuration ω^i is obtained by removing the $|{\circ\atop\bullet}|$-column on the right-hand side of wall i.
- Case ${\bullet\atop ?}\|{\circ\atop\circ}$. The reduced configuration ω^i is obtained by removing the two particles forming the ${}^{\bullet}|_{\circ}$-diagonal around wall i.
- Cases ${\bullet\atop\circ}\|{\times\atop\times}$ and ${\bullet\atop\circ}\|$. The reduced configuration ω^i is obtained by removing the $|{\bullet\atop\circ}|$-column on the left-hand side of wall i.
- Cases $\|{\times\atop\times}$ and ${\times\atop\times}\|$. The reduced configuration is obtained by removing the $|{\times\atop\times}|$-column on the border.

Lemma 6.1. *Let $\tilde{\omega}$ be a configuration of Ω_{n-1}. Let $S(\tilde{\omega})$ be the set of pairs (ω,i) of $\Omega_n\times\{0,\ldots,n\}$ having $\tilde{\omega}$ as reduced configuration,* i.e. *such that $\omega^i=\tilde{\omega}$. In particular let ω_0 be the configuration $|{\times\atop\times}|\tilde{\omega}$ and ω_n be the configuration $\tilde{\omega}|{\times\atop\times}|$, and define $S^0(\tilde{\omega})=S(\tilde{\omega})\setminus\{(\omega_0,0),(\omega_n,n)\}$. Then:*

- *The set $S^0(\tilde{\omega})$ is a cyclic orbit of $\bar{T}_1$: given $(\omega,i)\in S^0(\tilde{\omega})$, all other elements of $S^0(\tilde{\omega})$ can be reached by successive applications of $\bar{T}_1$.*
- *The set $S(\tilde{\omega})$ is a cyclic orbit of $\bar{T}_2$.*
- *If $\tilde{\omega}\in\Omega^0_{n-1}$ then $S^0(\tilde{\omega})\subset\Omega^0_n$ and $S^0(\tilde{\omega})$ is a cyclic orbit of $\bar{T}$.*

Proof. As can be checked on Figures 9 and 17, starting from a pair (ω,i) of the corresponding classes and iterating $\bar{T}_1$, $\bar{T}_2$ or $\bar{T}$, the selected wall moves to the left with the pair of black and white particles, and successively stops on the right-hand side of every black or $\times$ particle of the top row, until it reaches the left border. Similarly, as can be checked on Figures 10 and 18, iterating $\bar{T}_1$, $\bar{T}_2$ or $\bar{T}$ from a pair (ω,i) of the corresponding classes, the selected wall moves to the right with the pair of black and white particles, stopping on the left-hand side of every white and $\times$ particle of the top row, until it reaches the right border.

As shown by Figures 19–20, the application $\bar{T}_2$, and the applications $\bar{T}_1$ or $\bar{T}$ behave differently when the border is reached: $\bar{T}_2$ visits the configurations ω_0 or ω_n while $\bar{T}_1$ or $\bar{T}$ skips them and immediately restart moving in the opposite direction.

Starting from an element (ω,i) all other elements of $S(\tilde{\omega})$ (respectively $S^0(\tilde{\omega})$) are thus visited in a cycle by successive applications of $\bar{T}_2$ (respectively $\bar{T}_1$ or $\bar{T}$). □

Lemma 6.1 provides us with cycles in the transition graph on Ω_n, and each cycle is associated to a reduced configuration of Ω_{n-1}. The next lemma transports transitions from Ω_{n-1} to Ω_n.

Lemma 6.2. *Let $(\tilde{\omega}',j)=\bar{T}_1(\tilde{\omega},i)$ be a transition between two configurations of Ω_{n-1}. Then there exist k,i_+,j_+ and ω,ω' such that $(\omega,k)\in S(\tilde{\omega})$, $(\omega',k)\in S(\tilde{\omega})$, and $(\omega',j_+)=\bar{T}_1(\omega,i_+)$. The same holds for $\bar{T}_2$.*

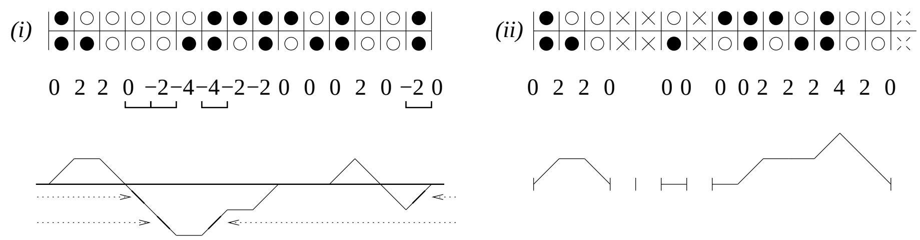

FIGURE 22. From (i) an element of $\overline{\Gamma}_{n+1}$, to (ii) one of Ω_n. The $(B(j)-W(j))_{j=0..n+1}$ are given under both configurations and graphically represented.

Proof. For $\bar{T}_1$ observe that in each case of Figure 17, and on the second leftmost case of Figure 19, a $|{}^{\circ}_{\bullet}|$-column can be inserted on the left border without interfering with the action of $\bar{T}_1$: take $\omega = |{}^{\circ}_{\bullet}|\tilde{\omega}$, $\omega' = |{}^{\circ}_{\bullet}|\tilde{\omega}$, $k = 0$, $i_+ = i+1$, $j_+ = j+1$. Similarly in each case of Figure 18, and on the leftmost case of Figure 19, a $|{}^{\bullet}_{\circ}|$-column can be inserted on the right border without interfering with the action of $\bar{T}_1$: take $\omega = \tilde{\omega}|{}^{\bullet}_{\circ}|$, $\omega' = \tilde{\omega}|{}^{\bullet}_{\circ}|$, $k = n$, $i_+ = i$, $j_+ = j$.

For $\bar{T}_2$ observe that in each case of Figures 18–20, an $|{}^{\times}_{\times}|$-column can be inserted, either on the left or on the right border, without interfering with the action of $\bar{T}_2$. □

Lemma 6.2 gives a transition between an element of the cycle associated to $\tilde{\omega}$ and an element of the cycle associated to $\tilde{\omega}'$. Taking the connectivity of the transition graph on Ω_{n-1} as induction hypothesis, we conclude that all cycles of Lemma 6.1 belong to the same connected component of the transition graph defined by $\bar{T}_2$ on Ω_n. Since every element of Ω_n belong to a cycle, this concludes the proof of the irreducibility of X.

As opposed to this the transition graph defined by $\bar{T}_1$ is seen to connect only configurations with the same number of $|{}^{\times}_{\times}|$-columns. In particular the chain $X_{\alpha\beta\gamma 0}$ with $\varepsilon = 0$ is not irreducible, but, the transition graph defined by $\bar{T}$ (or $\bar{T}_1$) on Ω^0_n is connected and the chain $X^0_{\alpha\beta\gamma}$ is irreducible.

Finally the chain $\widehat{T}$ is seen to be irreducible in a similar manner as soon as there is at least one $|{}^{\times}_{\times}|$-column.

7. THE NUMBER OF COMPLETE CONFIGURATIONS AND THE CYCLE LEMMA

Lemma 4.1. *The cardinality of* Ω_n *is* $\frac{1}{2}\binom{2n+2}{n+1}$.

Proof. Let Γ_{n+1} be the set of (unconstrained) configurations of $n+1$ black and $n+1$ white particles distributed between two rows of $n+1$ cells, so that $|\Gamma_{n+1}| = \binom{2n+2}{n+1}$. Among these configurations, we restrict our attention to the subset $\overline{\Gamma}_{n+1}$ of those ending with $|{}^{\bullet}_{\circ}|$- or a $|{}^{\bullet}_{\bullet}|$-column. Exchanging $\bullet$ and $\circ$ particles is a bijection between $\overline{\Gamma}_{n+1}$ and its complement in Γ_{n+1}, so that $|\overline{\Gamma}_{n+1}| = \frac{1}{2}\binom{2n+2}{n+1}$.

The proof of the lemma consists in a bijection ϕ between Ω_n and $\overline{\Gamma}_{n+1}$ (see Figure 22). Given $\omega \in \Omega_n$, its image $\phi(\omega)$ is obtained as follows: First, if the number of $|{}^{\times}_{\times}|$-columns of ω is even, add a $|{}^{\bullet}_{\circ}|$-column at the end of ω, otherwise add to it an $|{}^{\times}_{\times}|$-column. Then replace the first half of the $|{}^{\times}_{\times}|$-columns by $|{}^{\circ}_{\circ}|$-columns, and the remaining half by $|{}^{\bullet}_{\bullet}|$-columns (from left to right). By construction the resulting $\phi(\omega)$ belongs to $\overline{\Gamma}_{n+1}$. Conversely, consider $\gamma \in \overline{\Gamma}_{n+1}$, and let $d = \min(E(j))$ be the *depth* of γ. Then set $j_i = \min\{j \mid E(j) = -2i\}$, and $j'_i = \max\{j \mid E(j-1) = -2i\}$, for $i = 1, \ldots, \lfloor d/2 \rfloor$, and define the application ψ that first changes columns j_i and j'_i into $|{}^{\times}_{\times}|$-columns for all $i = 1, \ldots, \lfloor d/2 \rfloor$, and then removes the last column. By construction the blocks between two of the modified columns of γ satisfy the positivity condition, so that $\psi(\gamma) \in \Omega_{n+1}$. Finally the applications ϕ and ψ are clearly inverse of each other. □

Lemmas 2.2 and 4.2. *For any* $k+\ell+m = n$, *the cardinality of the set* $\Omega^{\ell}_{k,m}$ *of complete configurations with with* ℓ $|{}^{\times}_{\times}|$*-columns,* k *black and* m *white particles in the top row is* $\frac{\ell+1}{n+1}\binom{n+1}{k}\binom{n+1}{m}$.

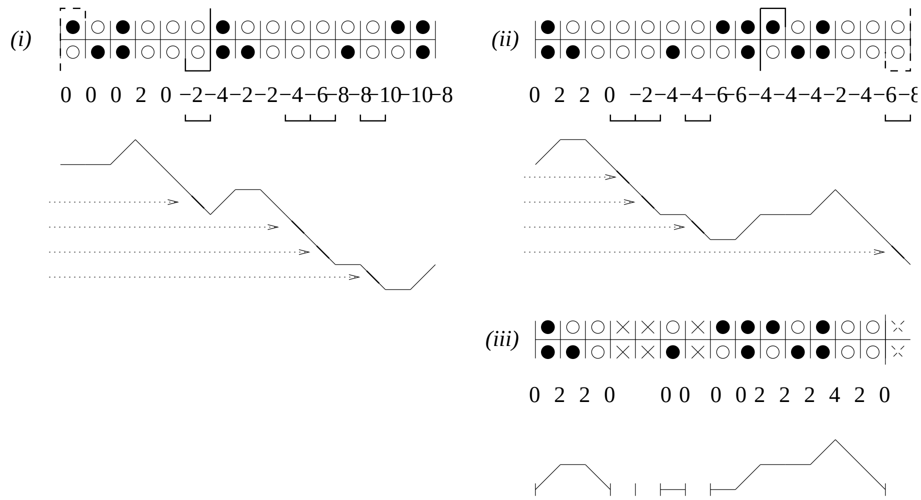

FIGURE 23. (i) An element of $\bar{\Delta}^{\ell+1}_{k,m}$ (with $\ell = 3$ and column $j = 6$ colored), (ii) its conjugate (with column $n+1-j$ colored), and (iii) the corresponding element of $\Omega^{\ell}_{k,m}$. The sequence $(B(j) - W(j))_{j=0..n+1}$ is given under each configuration and graphically represented.

Proof. The statement is verified using the cycle lemma (see [9, Ch. 11], or [11, Ch. 5]). Denote by $\Delta^{\ell+1}_n$ the set of configurations with $p = n - \ell = k + m$ black and $p + 2\ell + 2$ white particles distributed between two rows of $n+1$ cells. Then the cardinality of the subset $\Delta^{\ell+1}_{k,m}$ of elements of $\Delta^{\ell+1}_n$ that have k black particles in the top row and the other m in the bottom row is $\binom{n+1}{k}\binom{n+1}{m}$. In such a configuration the number of white particles exceeds by $2\ell + 2$ that of black particles, so that $E(n + 1) = -2\ell - 2$. Given ω in $\Delta^{\ell+1}_{k,m}$, let $d = \min(E(j))$ be the depth of ω, and set $j_i = \min\{j \mid E(j) = d+2i\}$, for $i = 0, \ldots, \ell$. By construction, these $\ell+1$ columns are $|^{\circ}_{\circ}|$-columns. On the one hand, let $\bar{\Delta}^{\ell+1}_{k,m}$ be the set of pairs (ω, j) where $\omega \in \Delta^{\ell+1}_{k,m}$ and $j \in \{j_0, \ldots, j_\ell\}$, so that $|\bar{\Delta}^{\ell+1}_{k,m}| = \binom{n+1}{k}\binom{n+1}{m} \cdot (\ell + 1)$. On the other hand, define the set $\bar{\Omega}^{\ell+1}_{k,m}$ of pairs (ω', i) where ω' is obtained from an element of $\Omega^{\ell}_{k,m}$ by adding a final $|^{\times}_{\times}|$-column, and $i \in \{0, \ldots, n\}$. By construction, $|\bar{\Omega}^{\ell+1}_{k,m}| = |\Omega^{\ell}_{k,m}| \cdot (n + 1)$.

The proof of the lemma consists in a bijection ϕ between $\bar{\Delta}^{\ell+1}_{k,m}$ and $\bar{\Omega}^{\ell+1}_{k,m}$ (see Figure 23). Given $(\omega, j) \in \bar{\Delta}^{\ell+1}_{k,m}$, let ω_1 denote the first j columns of ω, and ω_2 the $n+1-j$ others. Then by construction of j, the concatenation $\omega_2|\omega_1$ satisfies $E(i) > -2\ell-2$ for $i = 1, \ldots, n$, and $E(n+1) = -2\ell-2$. This implies that $\omega_2|\omega_1$ decomposes as a sequence $\omega'_0, \omega'_1, \ldots, \omega'_\ell$ of $\ell+1$ (possibly empty) blocks that satisfy the positivity constraint, each followed by a $|^{\circ}_{\circ}|$-column. Let ω' be obtained by replacing these $\ell + 1$ $|^{\circ}_{\circ}|$-columns by $|^{\times}_{\times}|$-columns. Then the map $(\omega, j) \to (\omega', n + 1 - j)$ is a bijection of $\bar{\Delta}^{\ell+1}_{k,m}$ onto $\bar{\Omega}^{\ell+1}_{k,m}$: the inverse bijection is readily obtained by first replacing the $|^{\times}_{\times}|$-columns into $|^{\circ}_{\circ}|$-columns, and then recovering the factorization $\omega_2|\omega_1$ from the fact that ω_2 has $n + 1 - j$ columns. $\square$

Lemmas 2.1 and 4.3. *The cardinality of the set Ω^{ℓ}_n of complete configurations of Ω_n that have ℓ $|^{\times}_{\times}|$-columns is $\frac{\ell+1}{n+1}\binom{2n+2}{n-\ell}$.*

Proof. The proof uses the same arguments than the proof of Lemma 4.2. The only difference is that, instead of counting elements of $\Delta^{\ell+1}_{k,m}$ with k black particles in the top row and m in the bottom row, we count elements of $\Delta^{\ell+1}_n$, the set of configurations of $n - \ell$ black particles and $n + 2 + \ell$ white particles distributed in two rows. Hence the previous factor $|\Delta^{\ell+1}_{k,m}| = \binom{n+1}{k}\binom{n+1}{m}$ is replaced by $|\Delta^{\ell+1}_n| = \binom{2n+2}{n-\ell}$. $\square$

Lemma 5.1. *The number* $|\widehat{\Omega}^{\ell}_{k,m}|$ *of configurations of* $|\widehat{\Omega}_n|$ *having* ℓ $|{}^{\times}_{\times}|$*-columns,* k *black particles at the top, and* m *at the bottom is* $\binom{n}{k}\binom{n}{m}$.

Proof. Recall that $\Delta^{\ell}_{k,m}$ denotes configurations of length n with k black and $m+\ell$ white particles in the top row, and m black and $k+\ell$ white particles in the bottom row, so that $|\Delta^{\ell}_{k,m}| = \binom{n}{k}\binom{n}{m}$. In order to prove the statement of the lemma we show that $\Delta^{\ell}_{k,m}$ and $\widehat{\Omega}^{\ell}_{k,m}$ are in bijection. Let $\delta \in \Delta^{\ell}_{k,m}$, and consider its depth $d = \min(E(i))$ and the ℓ columns $j_i = \min\{j \mid E(j) = d + 2i\}$, $i = 0, \dots, \ell-1$, as in the proof of Lemma 2.1. By definition of these columns, the positivity condition is satisfied by each block between two of them. Moreover, by definition of j_0 and $j_{\ell-1}$, the positivity condition is also satisfied by the concatenation $\omega_\ell|\omega_0$ of the final block ω_ℓ and the initial block ω_0. Hence transforming the columns $j_0, \dots, j_\ell$ into $|{}^{\times}_{\times}|$-columns, and arranging the two rows in a circle by fusing walls 0 and n at the apex yields a configuration $\phi(\delta)$ of $\widehat{\Omega}^{\ell}_{k,m}$ (recall that these configurations are not considered up to rotation). Conversely, given ω in $\widehat{\Omega}^{\ell}_{k,m}$, a unique element δ of $\Delta^{\ell}_{k,m}$ such that $\phi(\delta) = \omega$ is obtained by opening at the apex and transforming $|{}^{\times}_{\times}|$-columns into $|{}^{\circ}_{\circ}|$-columns. □

8. Conclusions and relations to Brownian excursions

The starting point of this paper was a "combinatorial Ansatz": the stationary distribution of the two particle TASEP with boundaries can be expressed in terms of Catalan numbers hence should have a nice combinatorial interpretation. In our interpretation, configurations of the TASEP are completed by a (usually hidden) second row in which particles go back. In the most interesting case $\alpha = \beta = \gamma = 1$, the resulting chain has a uniform stationary distribution so that the probability of a given TASEP configuration just reflects the diversity of possible rows hidden below it.

We do not claim that our combinatorial interpretation is of any physical relevance. However, apart from explaining the "magical" occurrence of Catalan numbers in the problem, it sheds new light on the recent results of Derrida *et al.* [4] connecting the TASEP with Brownian excursion. More precisely, using explicit calculations, Derrida *et al.* show that the density of black particles in configurations of the two particle TASEP can be expressed in terms of a pair (e_t, b_t) of independent processes, a Brownian excursion e_t and a Brownian motion b_t. In our interpretation these two quantities appear at the discrete level, associated to each complete configuration ω of Ω^0_n:

- The role of the Brownian excursion for ω is played by the halved differences $e(i) = \frac{1}{2}(B(i) - W(i))$ between the number of black and white particles sitting on the left of wall i, for $i = 0, \dots, n$. By definition of complete configurations, $(e(i))_{i=0,\dots,n}$ is a discrete excursion, that is, $e(0) = e(n) = 0$, $e(i) \geq 0$ and $|e(i) - e(i-1)| \in \{0,1\}$, for $i = 0, \dots, n$.
- The role of the Brownian motion is played for ω by the differences $b(i) = B_{top}(i) - B_{bot}(i)$ between the number of black particles sitting in the top and in the bottom row, on the left of wall i, for $i = 0, \dots, n$. This quantity $(b(i))_{i=0,\dots,n}$ is a discrete walk, with $|b(i) - b(i-1)| \in \{0,1\}$ for $i = 0, \dots, n$.

Since $e(i) + b(i) = 2B_{top}(i) - i$, the functions e and b allow one to describe the cumulated number of black particles in the top row of a complete configuration. Accordingly, the density of black particles in a given segment (i,j) is $(B_{top}(j) - B_{top}(i))/(j-i) = \frac{1}{2} + \frac{e(j)-e(i)}{2(j-i)} + \frac{b(j)-b(i)}{2(j-i)}$. This is a discrete version of the quantity considered by Derrida *et al.* in [4].

Now the two walks $e(i)$ and $b(i)$ are correlated since one is stationary when the other is not, and vice-versa: $|e(i) - e(i-1)| + |b(i) - b(i-1)| = 1$. Given ω, let $I_e = \{\alpha_1 < \ldots < \alpha_p\}$ be the set of indices of $|{}^{\bullet}_{\bullet}|$- and $|{}^{\circ}_{\circ}|$-columns, and $I_b = \{\beta_1 < \ldots < \beta_q\}$ the set of indices of $|{}^{\bullet}_{\circ}|$- and $|{}^{\circ}_{\bullet}|$-columns $(p+q = n)$. Then the walk $e'(i) = e(\alpha_i) - e(\alpha_{i-1})$ is the excursion obtained from e by ignoring stationary steps, and the walk $b'(i) = b(\beta_i) - b(\beta_{i-1})$ is obtained from b in the same way. Conversely given a simple excursion e' of length p, a simple walk b' of length q and a subset I_e of $\{1, \dots, p+q\}$ of cardinality p, two correlated walks e and b, and thus a complete configuration ω can be uniquely reconstructed. The consequence of this discussion is that the uniform distribution on Ω^0_n corresponds to the uniform distribution of triples (I_e, e', b') where, given I_e, the processes e' and b' are independent.

A direct computation shows that in the large n limit, with probability exponentially close to 1, a random configuration ω is described by a pair (e', b') of walks of roughly equal lengths $n/2 + O(n^{1/2+\varepsilon})$. In particular, up to multiplicative constants, the normalized pairs $(\frac{e'(tn/2)}{n^{1/2}}, \frac{b'(tn/2)}{n^{1/2}})$ and $(\frac{e(tn)}{n^{1/2}}, \frac{b(tn)}{n^{1/2}})$ both converge to the same pair (e_t, b_t) of independent processes, with e_t a standard Brownian excursion and b_t a standard Brownian walk.

Another possible outcome of our approach could be an explicit construction of a continuum TASEP by taking the limit of the Markov chain X, viewed as a Markov chain on pairs of walks. An appealing way to give a geometric meaning to the transitions in the continuum limit could be to use a representation in terms of parallelogram polyominos [11], using the process $e(t)$ (or e_t in the continuum limit) to describe the width of the polymonino and the process $b(t)$ (or b_t in the continuum limit) to describe the vertical displacement of its spine.

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